Representations of Finite-Dimensional Lie Algebras

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1 Introdcution

One of the most prolific applications of representation theory is in quantum algebra, or the mathematics of quantum physics. In particular, the analysis of Lie groups and their associated algebras is of great interest to physicists, and representations play a significant role in understanding these objects, such as in [2]. Recent research by Marcolli, Chomksy, and Berwick [3] has connected quantum theory to Noam Chomsky's linguistic theory of Minimalist Grammar. This surprising connection between linguistics and quantum physics is what drew my attention to quantum algebra, and serves as the motivation for this project.

The purpose of this project is to provide an accessible introduction to Lie groups and their algebras. We will follow Hall in [1] by extending our theory of representations to special types of infinite groups. Many proofs end up looking similar to those by Hall, though I tried to be more thorough about which results I was using. All solutions and boxed results which have an equivalent in [1] have references at the beginning.

2 Worksheets

1. Matrix Lie groups and their representations

Definition. We can define a norm on $M_n(\mathbb{C})$ by taking the maximum of its entries. This is equivalent to the sup-norm when viewing $M_n(\mathbb{C})$ as $\mathbb{C}^{n^2}$. A sequence of $n\times n$ matrices (A_m) converges to A if for every $\epsilon>0$ there exists an $N\in\mathbb{N}$ such $\|A_m - A\| < \epsilon$ for all $m\geq N$.

Definition (1.4)**.** Let $G\leq \mathrm{GL}(\mathbb{C}^n)$ be a subgroup of $n\times n$ matrices. We say that G is a *matrix Lie group* if it contains all of its *invertible* limit points. That is, for every convergent sequence $A_m\to A$ in G, either $A\in G$ or A is not invertible.

A. 1. Show that $\{I\}$ and $\mathrm{GL}(\mathbb{C}^n)$ are matrix Lie groups of $\mathrm{GL}(\mathbb{C}^n)$ (c.f. §1.2).

Solution: Both $\{I\}$ and $\mathrm{GL}(\mathbb{C}^n)$ are subgroups of $\mathrm{GL}(\mathbb{C}^n)$. The singleton set $\{I\}$ is closed with respect to $\mathrm{GL}(\mathbb{C}^n)$ since it is finite. If $A_m\to A$ in $\mathrm{GL}(\mathbb{C}^n)$, then either $A\in\mathrm{GL}(\mathbb{C}^n)$ or $A\notin\mathrm{GL}(\mathbb{C}^n)$. In either case, $\mathrm{GL}(\mathbb{C}^n)$ contains its invertible limit points.

2. Prove or disprove: $\{\lambda I : \lambda\in\mathbb{C}^\times\}$ is a matrix Lie group of $\mathrm{GL}(\mathbb{C}^n)$.

Solution: $\mathbb{C}^\times$ is a group under multiplication, and so $G := \{\lambda I : \lambda\in\mathbb{C}^\times\}$ is also. Suppose $(\lambda_n I)\to\lambda I$ for some $\lambda\in\mathbb{C}$. If $\lambda\neq 0$, then $\lambda I\in G$. If $\lambda=0$, then $\lambda I = 0\notin\mathrm{GL}(\mathbb{C}^n)$. In either case, G is closed in $\mathrm{GL}(\mathbb{C}^n)$.

3. Assuming that the determinant is a continuous function $\mathrm{GL}(\mathbb{C}^n)\to\mathbb{C}$, prove that the *special linear group* $\mathrm{SL}(\mathbb{C}^n)$ of transformations with determinant 1 is a matrix Lie group of $\mathrm{GL}(\mathbb{C}^n)$.

Solution: First notice that $\mathrm{SL}(\mathbb{C}^n)$ is closed under multiplication and inverses since the determinant is multiplicative. The determinant of the identity is 1, and so $\mathrm{SL}(\mathbb{C}^n)$ is a multiplicative subgroup of $\mathrm{GL}(\mathbb{C}^n)$. Suppose $A_m\to A$, where $A_m\in\mathrm{SL}(\mathbb{C}^n)$ for all m. Since the determinant is a continuous function on $\mathrm{GL}(\mathbb{C}^n)$, it follows that $\lim_{m\to\infty}\det(A_m) = \det(A) = 1$. Thus $\mathrm{SL}(\mathbb{C}^n)$ is topologically closed also.

4. Assuming for fixed $v,w\in\mathbb{C}^n$ that the map $A\mapsto\langle Av, Aw\rangle$ is continuous on $\mathrm{GL}(\mathbb{C}^n)$, prove that the group $U(\mathbb{C}^n)$ of unitary matrices is a matrix Lie group.

Solution: Notice first that $I \in U(\mathbb{C}^n)$ since $\langle Iv, Iw\rangle = \langle v,w\rangle$ for all $v,w \in \mathbb{C}^n$ trivially. If $A,B \in U(\mathbb{C}^n)$, then $\langle v,w\rangle = \langle Bv,Bw\rangle = \langle ABv,ABw\rangle$, and so $AB \in U(\mathbb{C}^n)$. Finally, if $A \in U(\mathbb{C}^n)$, then $\langle A^{-1}v,A^{-1}w\rangle = \langle AA^{-1}v,AA^{-1}w\rangle = \langle v,w\rangle$, so $A^{-1} \in U(\mathbb{C}^n)$. Thus $U(\mathbb{C}^n)$ is a multiplicative subgroup of $\mathrm{GL}(\mathbb{C}^n)$.

Suppose $A_m \to A$ in $U(\mathbb{C}^n)$, where $A \in \mathrm{GL}(\mathbb{C}^n)$. Fixing $v,w \in \mathbb{C}^n$, the continuity of the map $A \mapsto \langle Av,Aw\rangle$ implies the functional limit $\lim_{m\to\infty}\langle A_m v,A_m w\rangle = \langle Av,Aw\rangle$, but $\langle A_m v,A_m w\rangle = \langle v,w\rangle$ for all $m \in \mathbb{N}$, and so $\langle Av,Aw\rangle = \langle v,w\rangle$. Thus $A \in U(\mathbb{C}^n)$ also. Hence $U(\mathbb{C}^n)$ is a topologically closed subgroup of $\mathrm{GL}(\mathbb{C}^n)$, and so it is a matrix Lie group.

5. Consider a slight variation on $\mathrm{SL}(\mathbb{C}^n)$: the set S of 2×2 matrices A such that $0<\det(A)<1$, union the identity matrix I.

Solution: $S\backslash I$ is closed under multiplication since the determinant is multiplicative and the product of two numbers in (0,1) is in (0,1). The product of I and any other matrix in S has determinant less than 1, and the product of I and itself is I. Thus S is a multiplicative subgroup of $\mathrm{GL}(\mathbb{C}^2)$. However, consider the matrices $A_m = \begin{pmatrix} 0 & \frac{1}{m}-1 \\ 1-\frac{1}{m} & 0 \end{pmatrix}$. The determinant of A_m is $\left(\frac{m-1}{m}\right)^2$, and so $A_m \in S$ for all m, but A_m approaches $A := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ as $m \to \infty$. A is in the general linear group, but has determinant 1 and so it is not in S. Thus S is not topologically closed in $\mathrm{GL}(\mathbb{C}^n)$, and S is not a matrix Lie group.

Definition (1.18)**.** Let $G \leq \mathrm{GL}(\mathbb{C}^n)$ and $H \leq \mathrm{GL}(\mathbb{C}^m)$ be matrix Lie groups. A *Lie group homomorphism* $\Phi\colon G \to H$ is a *continuous* group homomorphism from G to H.)

Definition (4.1)**.** Let G be a matrix Lie group and V a finite dimensional vector space. A *finite-dimensional representation* of G is a Lie group homomorphism $\Phi\colon G \to \mathrm{GL}(V)$.

B. (c.f. §4.2). Let $G \leq \mathrm{GL}(\mathbb{C}^n)$ be a matrix Lie group.

1. Show that the trivial representation $\Phi\colon G \to \mathrm{GL}(\mathbb{C}^n)$ given by $\Phi(A) = I$ is a representation.

Solution: Suppose $A,B \in G$, notice first that $\Phi(A)\Phi(B) = I = \Phi(AB)$. If $\epsilon > 0$, then
$$\|\Phi(A) - \Phi(B)\| = \|0\| = 0 < \epsilon.$$

2. Show that the inclusion $\iota\colon G \to \mathrm{GL}(\mathbb{C}^n)$ given by $\iota(A) = A$ is a represntation.

Solution: We have that $\iota(A)\iota(B) = AB = \iota(AB)$ by closure of G under multiplication. If $\epsilon > 0$, notice that $\|\iota(A) - \iota(B)\| < \epsilon$ implies $\|A - B\| < \epsilon$ by definition.

3. Disprove: For all matrix Lie groups G, $\Psi\colon G \to \mathrm{GL}(\mathbb{C}^n)$ which maps $\Psi(A) = A^{-1}$ is a representation.

Solution: If Ψ were a representation, then for all $A,B \in G$.
$$AB = \Psi(B^{-1}A^{-1}) = \Psi(B^{-1})\Psi(A^{-1}) = BA.$$
Thus $AB = BA$. This is not true in general, and so Ψ is not a representation in general.

4. Are there any matrix Lie groups G for which Ψ is a representation?

Solution: From the previous proof, we have that Ψ is a representation if and only if G is commutative. An example is $\{\lambda I | \lambda \in \mathbb{C}^\times\}$.

Definition. A matrix Lie group G is *compact* if it is closed and bounded in $\mathrm{GL}(\mathbb{C}^n)$ with respect to the sup-norm metric on $M_n(\mathbb{C})$. Since every matrix Lie group is closed in $(\mathbb{C}^n)$, we can equivalently say a matrix Lie group is compact if it is bounded.

The following comes from the theory of *differential forms* (Appendix B in [1]):

Proposition (c.f. Theorem 4.2.8). *If G is a* compact *matrix Lie group and $f: G \to \mathbb{C}$ is differentiable, we may define a map $\alpha: G \to \mathbb{R}_{>0}$ such that $\int_G f(A)\alpha(A)$ converges and*

$$\int_G f(A)\alpha(A) = \int_G f(AB)\alpha(A)$$

for all $B \in G$.

C. In this part, we will show Maschke's theorem for finite-dimensional representations on compact matrix Lie groups.

1. We have for finite groups that every representation is equivalent to a unitary representation. Why can't we use the same argument for infinite groups?

Solution: Proposition 3.2.4 in [4] is the statement for finite groups. If φ is our representation, we want to find an inner product for which φ is unitary. We construct an "averaging" inner product

$$(v,w) = \sum_{g \in G} \langle \varphi(g)v, \varphi(g)w \rangle$$

which we can show is unitary by change of variable. We then construct an equivalence $\varphi \sim \rho$ so the standard inner product acts on ρ like our constructed inner product acts on φ.

However, when the group is infinite, the sum is no longer well defined.

2. Let G be a compact matrix Lie group, and $\Phi: G \to \mathrm{GL}(V)$ a finite-dimensional representation. Assume without proof that the function $(v,w) := \int_G \langle \Phi(A)v, \Phi(A)w \rangle \alpha(A)$ converges for some $\alpha: G \to \mathbb{R}_{>0}$.

   a. Prove that $(\cdot,\cdot)$ is an inner product on V.

   Solution: The function $(\cdot,\cdot)$ inherits linearity and conjugate symmetry from the standard product and the construction of an integral as a sum. Positive-definiteness derives from positive-definiteness of the standard product and that $\alpha(A) > 0$ for all $A \in G$.

   b. Prove that Φ is unitary with respect to $(\cdot,\cdot)$.

   Solution: If we fix $B \in G$, it follows that

$$\begin{aligned}(\Phi(B)v, \Phi(B)w) &= \int_G \langle \Phi(A)\Phi(B)v, \Phi(A)\Phi(B)w \rangle \alpha(\Phi(A)) \\ &= \int_G \langle \Phi(AB)v, \Phi(AB)w \rangle \alpha(\Phi(A)) \\ &= \int_G \langle \Phi(A)v, \Phi(A)w \rangle \alpha(\Phi(A)) \\ &= (v,w).\end{aligned}$$

3. Look through the results proven in 3.1 and 3.2 in Steinberg [4]. Which results require the finiteness of G? The finite-dimensionality of V?

Solution: Lemma 3.1.19 explicitly requires $\dim(V) = 2$, but it is not used for future results. Proposition 3.2.4 requires the finiteness of G. Thus Corollary 3.2.5 and Maschke's Theorem require the finiteness of G. Maschke's explicitly requires V to be finite-dimensional, since it uses induction on $\dim(V)$.

4. Suppose G is a compact matrix Lie group and $\Phi\colon G \to \mathrm{GL}(V)$ is a finite-dimensional representation. Discuss how we can adapt results in Steinberg [4] to show Φ is completely reducible.

Solution: By C2, Φ is equivalent to a unitary representation. We can then prove Corollary 3.2.5 for Φ, since the only result it uses requiring the finitness of G is Proposition 3.2.4. We may then recreate the proof of Theorem 3.2.8 (Maschke's Theorem) without issue.

2. Lie algebras

Definition (3.1)**.** Let $\mathbb{F}$ be a field. A *Lie algebra* over a field $\mathbb{F}$ is an $\mathbb{F}$-vector space $\mathfrak{g}$ equipped with a bracket map (the product over $\mathfrak{g}$) denoted $[\cdot,\cdot]\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ such that
1. (Bilinearity): $c[X,Y] = [cX,Y] = [X,cY]$ for all $X,Y \in \mathfrak{g}$, $c \in \mathbb{F}$.
2. (Skew-symmetry): $[Y,X] = -[X,Y]$ for all $X,Y \in \mathfrak{g}$
3. (Idempotence): $[X,X] = 0$ for all $X \in \mathfrak{g}$
4. (The Jacobi Identity): $[X,[Y,Z]] + [Y,[X,Z]] + [Z,[X,Y]] = 0$ for all $X,Y,Z \in \mathfrak{g}$.

A. 1. Which part of Definition 16.10 shows that a Lie algebra is indeed an $\mathbb{F}$-algebra?

Solution: Condition 1 tells us that $[\cdot,\cdot]$ is bilinear, and an $\mathbb{F}$-algebra is a vector space (in this case, $\mathfrak{g}$) equipped with a bilinear map $\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$.

2. Suppose $\mathfrak{g}$ is a vector-subspace of an associative $\mathbb{F}$-algebra $\mathcal{A}$. Define $[\cdot,\cdot] : \mathcal{A}\times\mathcal{A}\to\mathcal{A}$ by $[X,Y] = XY - YX$, and suppose $\mathfrak{g}$ is closed under this operation..

a. Show that $[\cdot,\cdot]$ is bilinear.

Solution: Fix $X,Y \in \mathcal{A}$ and $c \in \mathbb{F}$. Then, by bilinearity of the product in $\mathcal{A}$, we have

$$\begin{aligned} c[X,Y] &= c(XY - YX) \\ &= (cX)Y - Y(cX) \\ &= [cX,Y] \\ &= X(cY) - (cY)X \\ &= [X,cY]. \end{aligned}$$

b. Show that $[Y,X] = -[X,Y]$ for $X,Y \in \mathcal{A}$.

Solution: By distributivity in $\mathcal{A}$,

$$\begin{aligned} [Y,X] &= YX - XY \\ &= -(XY - YX) \\ &= -[X,Y]. \end{aligned}$$

c. Conclude that $(\mathfrak{g},[\cdot,\cdot])$ is a Lie algebra. (c.f. Example 3.3)*

Solution: We have bilinearity and skew-symmetry from (a) and (b), respectively. By definition of the bracket, we obtain idempotence:
$$[X,X]=XX-XX=0.$$
Finally, since $\mathcal{A}$ is associative, we have the Jacobi identity
$$\begin{aligned}[X,[Y,Z]]+[Y,[X,Z]]+[Z,[X,Y]] &= X(YZ-ZY)-(YZ-ZY)X\\ &\quad +Y(XZ-ZX)-(XZ-ZX)Y\\ &\quad +Z(XY-YX)-(XY-YX)Z\\ &= X(YZ)-(XY)Z+X(ZY)-(XZ)Y\\ &\quad +Y(XZ)-(YX)Z+Y(ZX)-(YZ)X\\ &\quad +Z(XY)-(ZX)Y+Z(YX)-(ZY)X\\ &=0.\end{aligned}$$
Thus, $\mathfrak{g}$ is a vector space equipped with a bilinear bracket map (the restriction of $[\cdot,\cdot]$ to $\mathfrak{g}$) satisfying all properties of Definition 3.1.

Definition. Let $X\in M_n(\mathbb{C})$ be a matrix. We define the *matrix exponential* e^X as the power series
$$e^X=\sum_{m=0}^{\infty}\frac{1}{m!}X^m.$$

Proposition (2.1)**.** *The sum* e^X *converges for all* $X\in M_n(\mathbb{C})$.

Proposition. *If* $\sum a_m X^m$ *converges, then so does* $\sum a_m T\circ X^m$ *and is equal to* $T\circ\sum a_m X^m$.

B. 1. Let $A\in M_n(\mathbb{C})$ be a diagonal matrix with entries $A_{i,i}=\lambda_i$. Show that e^A is a diagonal matrix with entries $(e^A)_{i,i}=e^{\lambda_i}$. *Hint: consider the linear transformation* $P_{i,j}\colon \mathbb{C}^n\to\mathbb{C}$ *which maps* A *to* $A_{i,j}$.

Solution: Notice first that A^m is diagonal with entries $(A^m)_{i,i}=\lambda_i^m$. Let $P_{i,j}\colon M_n(\mathbb{C})\to\mathbb{C}$ be the linear transformation which maps $A\mapsto A_{i,j}$. Fix $1\le i\le n$. Then by linearity
$$\begin{aligned}P_{i,i}(e^A) &= P_{i,i}\circ\left(\sum_{m=0}^{\infty}\frac{1}{m!}A^m\right)\\ &=\sum_{m=0}^{\infty}\frac{1}{m!}P_{i,i}\circ A^m\\ &=\sum_{m=0}^{\infty}\frac{\lambda_i^m}{m!}\\ &=e^{\lambda_i}\end{aligned}$$
A similar argument shows $P_{i,j}(e^A)=0$ for $1\le i\ne j\le n$. Thus e^A is diagonal with entries $(e^A)_{i,i}=e^{\lambda_i}$.

2. Show that $e^{TXT^{-1}}=Te^XT^{-1}$ for all $T\in\mathrm{GL}(C^n)$ and $X\in M_n(\mathbb{C})$ (c.f. Proposition 2.3 no. 6)

*Aside: Echoing the words of Hall, the Jacobi identity "means that the bracket operation *behaves* as if it were $XY-YX$ in some associative algebra, even if it were not defined that way." [1, p. 50] It can be proven that every Lie algebra derives from some associative algebra using the bracket $[X,Y]=XY-YX$, and this is why we use the bracket notation as opposed to the standard product notation.

Solution: Fix $X\in M_n(\mathbb{C})$ and $T\in \mathrm{GL}(C^n)$. Notice that
$$(TXT^{-1})^m = TXT^{-1}\circ TXT^{-1}\circ\cdots\circ TXT^{-1} = TX^mT^{-1}.$$
Thus by distributivity of matrix multiplication,
$$\begin{aligned} e^{TXT^{-1}} &= \sum_{m=0}^{\infty}\frac{1}{m!}(TXT^{-1})^m \\ &= \sum_{m=0}^{\infty} T\circ\left(\frac{1}{m!}X^m\right)\circ T^{-1} \\ &= T\circ\left(\sum_{m=0}^{\infty}\frac{1}{m!}X^m\right)\circ T^{-1} \\ &= Te^XT^{-1}. \end{aligned}$$

3. Show that $e^{sX}e^{tX}=e^{(s+t)X}$ for all $s,t\in\mathbb{R}$ (c.f. Proposition 2.3 no. 4). *Hint: you need to use the Cauchy product.*

Solution: I had to see the book's use of the Cauchy product before I could solve this myself. Thankfully, it's pretty easy from there. We can use the Cauchy product and binomial theorem to obtain
$$\begin{aligned} e^{sX}e^{tX} &= \left(\sum_{m=0}^{\infty}\frac{s^m}{m!}X^m\right)\left(\sum_{m=0}^{\infty}\frac{t^m}{m!}X^m\right) \\ &= \sum_{m=0}^{\infty}\left(\sum_{k=0}^{m}\frac{s^k}{k!}\frac{t^{m-k}}{(m-k)!}\right)X^m \\ &= \sum_{m=0}^{\infty}\left(\sum_{k=0}^{m}\frac{1}{m!}\binom{m}{k}s^kt^{m-k}\right)X^m \\ &= \sum_{m=0}^{\infty}\frac{1}{m!}(s+t)^mX^m \\ &= e^{(s+t)X} \end{aligned}$$

Theorem (2.11, Lie product formula). *Suppose $X,Y\in M_n(\mathbb{C})$, then*
$$e^{X+Y} = \lim_{m\to\infty}(e^{X/m}e^{Y/m})^m$$

Definition (3.18). If $G\leq \mathrm{GL}(\mathbb{C}^n)$ is a matrix Lie group, then the *Lie algebra* $\mathfrak{g}$ of G is the set
$$\mathfrak{g}=\{X\in M_n(\mathbb{C})\,|\,e^{tX}\in G \text{ for all } t\in\mathbb{R}\}$$

C. Let G be a matrix Lie group. Let $\mathfrak{g}$ be the Lie algebra of G. In this part, we will prove $\mathfrak{g}$ is indeed a Lie algebra.

1. Prove that for all $X\in M_n(\mathbb{C})$, e^X is invertible with inverse e^{-X} (c.f. Proposition 2.3 no. 3)

Solution: By B3 we have that
$$e^Xe^{-X}=e^{(1-1)X}=e^0=I$$
and similarly we have $e^{-X}e^X=I$.

2. Prove that if $X,Y\in\mathfrak{g}$ and $r,s\in\mathbb{R}$, then $rX+sY\in\mathfrak{g}$. This implies that $\mathfrak{g}$ is a closed subset of $M_n(\mathbb{C})$ (c.f. Theorem 3.20 no. 2, 3).

Solution: I needed to see the book's use of the Lie product formula to solve this. Fix $X,Y \in \mathfrak{g}$ and $r,s \in \mathbb{R}$. To show $rX+sY \in \mathfrak{g}$, fix $t \in \mathbb{R}$. Then by the Lie product formula,

$$e^{t(rX+sY)} = e^{rtX+stY} = \lim_{m\to\infty} (e^{\frac{rt}{m}X} e^{\frac{st}{m}Y})^m.$$

But $e^{\frac{rt}{m}X}$ and $e^{\frac{st}{m}Y}$ are in G for all m, so by the closure of G under multiplication, $(e^{\frac{rt}{m}X} e^{\frac{st}{m}})^m$ is in G for all m. Since G is closed in $\mathrm{GL}(\mathbb{C}^n)$, and $e^{t(rX+sY)}$ is invertible by C1, $e^{t(rX+sY)}$ is in G. Since t was arbitrary, we have that $rX+sY \in \mathfrak{g}$.

3. Assume that the product formula holds for differentiable matrix-valued functions $\mathbb{R} \to M_n(\mathbb{C})$. Fix $X,Y \in \mathfrak{g}$. Prove that the derivative $\frac{\mathrm{d}}{\mathrm{d}t} e^{tX} Y e^{-tX}$ at $t=0$ is equal to $[X,Y]$.

Solution: While it was clear to me we needed to prove $\mathfrak{g}$ was a subalgebra of $M_n(\mathbb{C})$, the use of a product rule and conjugation identity is probably the most difficult logical step of this project. Needless to say I needed to reference the book's solution for this problem.

By the product formula, we can compute

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} e^{tX} Y e^{-tX} &= \frac{\mathrm{d}}{\mathrm{d}t}(e^{tX}Y)e^{-tX} + Y\frac{\mathrm{d}}{\mathrm{d}t}(e^{-tX})e^{tX} \\ &= Xe^{tX}Ye^{-tX} - e^{tX}YXe^{-tX}. \end{aligned}$$

When $t=0$, this derivative is equal to $XY-YX$, as desired.

4. Conclude that $[X,Y] \in \mathfrak{g}$ for all $X,Y \in \mathfrak{g}$ (c.f. Theorem 3.20 no. 4). *Hint: use the limit definition of the derivative, and notice $e^{tX}Ye^{-tX}$ is a conjugation.*

Solution: Fix $X,Y \in \mathfrak{g}$. First notice that $e^{tX}Ye^{-tX} \in \mathfrak{g}$ since

$$e^{s(e^{tX}Ye^{-tX})} = e^{tX}e^{sY}e^{-tX}$$

by B2 and G is closed under multiplication. By C3 we have that

$$[X,Y] = \frac{\mathrm{d}}{\mathrm{d}t} e^{tX}Ye^{-tX} = \lim_{t\to 0} \frac{e^{tX}Ye^{-tX} - Y}{t}$$

By C2, $(e^{tX}Ye^{-tX} - Y)/t$ is in $\mathfrak{g}$ for all $t \in \mathbb{R}\setminus\{0\}$. Also by C2, the closure of $\mathfrak{g}$ implies the limit $[X,Y]$ is also in $\mathfrak{g}$, as desired.

5. Discuss how you've proven $\mathfrak{g}$ is a Lie algebra.

Solution: $M_n(\mathbb{C})$ is an associative algebra under matrix multiplication, and we have proven $\mathfrak{g}$ is a subspace of $M_n(\mathbb{C})$ which is closed under the bracket map $[X,Y] = XY - YX$. Thus by A2, $\mathfrak{g}$ is a Lie algebra.

D. Prove that the Lie algebra of $\mathrm{GL}(\mathbb{C}^n)$ is $M_n(\mathbb{C})$. We will refer to this algebra as $\mathfrak{gl}(V)$ for any finite dimensional vector space V (c.f. Proposition 3.23).

Solution: Notice that the Lie algebra of $\mathrm{GL}(\mathbb{C}^n)$ must be a subset of $M_n(\mathbb{C})$ by definition, so it suffices to show that every matrix in $M_n(\mathbb{C})$ is in the Lie algebra of $\mathrm{GL}(\mathbb{C}^n)$. But e^{tX} is invertible for all $X \in M_n(\mathbb{C})$ by C1, and so $e^{tX} \in \mathrm{GL}(\mathbb{C}^n)$ for all $X \in M_n(\mathbb{C})$. Thus X belongs to the Lie algebra of $\mathrm{GL}(\mathbb{C}^n)$. Since X was arbitrary, $M_n(\mathbb{C})$ is the Lie algebra of $\mathrm{GL}(\mathbb{C}^n)$

3 Quiz

1. Complete the following definition. Let $\mathbb{F}$ be a field, $\mathfrak{g}$ a vector space over $\mathbb{F}$, and $[\cdot,\cdot]\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ a bilinear map. $[\cdot,\cdot]$ satisfies the *Jacobi identity* if...

> **Solution:** For all $X,Y,Z\in\mathfrak{g}$, we have
> $$[X,[Y,Z]]+[Y,[X,Z]]+[Z,[X,Y]]=0.$$

2. Give an example of a subgroup of $\mathrm{GL}(\mathbb{C}^n)$ which is *not* a matrix Lie group.

> **Solution:** 1.B4 ($\{T\in\mathrm{GL}(\mathbb{C}^n)\,|\,0<\det(T)<1\}\cup\{I\}$) is an example, though other examples exist.

3. Suppose $\varphi\colon G\to\mathrm{GL}(\mathbb{C}^n)$ is a representation of a finite abelian group G. Find $e^{\varphi(g)}$ for all $g\in G$, in terms of the irreducible components of φ.

> **Solution:** By Maschke's Theorem and Corollary 4.1.8, we have that φ is equivalent to the sum of n degree 1 irreducible representations $\varphi\sim\varphi^{(1)}\oplus\cdots\oplus\varphi^{(n)}$. Let $T\colon\mathbb{C}^n\to\mathbb{C}^n$ be the isomorphism which relates φ to this sum.
>
> Fix a $g\in G$. It follows that the matrix $A:=T^{-1}\varphi(g)T$ is diagonal with entries $A_{i,i}=\varphi^{(i)}(g)$. By B1 and B2, we have
> $$\begin{aligned} e^{\varphi(g)} &= e^{TAT^{-1}} \\ &= Te^{A}T^{-1} \\ &= T\begin{pmatrix} e^{\varphi^{(1)}(g)} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{\varphi^{(n)}(g)} \end{pmatrix}T^{-1}. \end{aligned}$$

4 Homework Problem

This problem requires some more heavy topological arguments, so I thought it'd work better for a longer-form homework problem. I was only able to solve the analytic arguments after seeing the book's process. In a more realistic teaching environment, I'd fragment the problem further and spend more time introducing calculus on Lie groups. However, I wanted to compromise and conclude the project with a familiar representation-theoretic result.

> **Definition** (3.6, 4.1). A *Lie algebra homomorphism* is a continuous, linear map $\varphi\colon\mathfrak{g}_1\to\mathfrak{g}_2$ such that $\varphi([X,Y])=[\varphi(X),\varphi(Y)]$ for all $X,Y\in\mathfrak{g}_1$. A *Lie algebra representation* is a Lie algebra homomorphism $\mathfrak{g}\to\mathsf{gl}(V)$.
>
> **Corollary** (3.47). *If G is a connected matrix Lie group, every element A of G van be written in the form*
> $$A=e^{X_1}e^{X_2}\dots e^{X_m}$$
> *for some $X_1,\dots,X_m\in\mathfrak{g}$.*

Proposition (4.4)**.** *Let* $\Phi\colon G \to \mathrm{GL}(V)$ *be a matrix Lie group representation. Then there exists a Lie algebra representation* $\varphi\colon \mathfrak{g} \to \mathsf{gl}(V)$ *such that*

$$\Phi(e^{tX}) = e^{t\varphi(X)}.$$

Moreover $\Phi(e^{tX})$ *is differentiable at* $t=0$ *and satisfies,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(e^{tX})\Big|_{t=0} = \varphi(X)$$

1. In this problem, we will prove an analogue of Maschke's theorem for Lie algebras. Suppose G is a *connected* matrix Lie group and $\mathfrak{g}$ its Lie algebra (c.f. Proposition 4.5). Let $\Phi\colon G \to \mathrm{GL}(\mathbb{C}^n)$ be a representation and let $\varphi\colon \mathfrak{g} \to \mathsf{gl}(\mathbb{C}^n)$ be its corresponding Lie algebra representation (as in 4.4).

   (a) Let V be a subspace of $\mathbb{C}^n$. Show that if V is closed under $\varphi(X)$, then V is closed under $e^{\varphi(X)}$.

   Solution: Suppose V is closed under $\varphi(X)$. Then V is closed under $\frac{1}{m!}\varphi(X)^m$ for all m. Fix $v \in V$ and consider the sequence $(v_k) = \left((\sum_{m=0}^{k}\varphi(X)^m/m!)v\right)$. Notice each $v_k \in V$ since

   $$\left(\sum_{m=0}^{k}\frac{1}{m!}\varphi(X)^m\right)v = \sum_{m=0}^{k}\frac{1}{m!}\varphi(X)^m v$$

   which is a linear combination of vectors in V. Since V is a topologically closed subspace of $\mathbb{C}^n$, it follows that the limit $e^{\varphi(X)}v$ is also in V. Since v was arbitrary, V is closed under $e^{\varphi(X)}$.

   (b) Show that if V is $\mathfrak{g}$-invariant with respect to φ, then V is G-invariant with respect to Φ.

   Solution: Suppose G is connected and V is $\mathfrak{g}$-invariant with respect to φ. To show V is G-invariant with respect to Φ, fix an arbitrary matrix $X \in G$. Since G is connected, we have by Corollary 3.47 that $X = e^{X_1}...e^{X_m}$ for $X_1,...,X_m \in \mathfrak{g}$. By Proposition 4.4 and (a) we have that V is closed under $e^{\varphi(X_i)} = \Phi(e^{X_i})$ for all $1 \le i \le m$. Thus V is closed under

   $$\Phi(e^{X_1})...\Phi(e^{X_m}) = \Phi(e^{X_1}...e^{X_m}) = \Phi(X).$$

   Since $X \in G$ was arbitrary, it follows that V is G-invariant with respect to Φ.

   (c) Conclude that if Φ is irreducible, then φ is irreducible.

   Solution: We prove the contrapositive. Suppose φ is not irreducible. Then there exists a nonzero, non-trivial vector space V which is $\mathfrak{g}$-invariant with respect to φ. By (b), V is G-invariant with respect to Φ, and it follows that Φ is not irreducible.

   (d) Show that if $\Phi \sim \Psi$, then $\varphi \sim \psi$, where ψ is the Lie algebra representation of Ψ. *Hint: differentiate* $\Phi(e^{tX})$ *where* $X \in \mathfrak{g}$.

   Solution: Let $\Psi\colon G \to \mathrm{GL}(\mathbb{C}^n)$ be a representation and $\psi\colon \mathfrak{g} \to \mathsf{gl}(\mathbb{C}^n)$ be its corresponding Lie algebra representation, and suppose $\Phi \sim \Psi$. Then there exists a map $T \in \mathrm{GL}(\mathbb{C}^n)$ such that $\Phi(X) = T^{-1}\Psi(X)T$ for all $X \in G$. In particular, $\Phi(e^{tX}) = T^{-1}\Psi(e^{tX})T$ for all $t \in \mathbb{R}$ and $X \in \mathfrak{g}$. Thus, differentiating both sides, we have for all $X \in \mathfrak{g}$ that

   $$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(e^{tX})\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}T^{-1}\Psi(e^{tX})T\Big|_{t=0} = T\frac{\mathrm{d}}{\mathrm{d}t}\Psi(e^{tX})\Big|_{t=0}T^{-1}.$$

   which shows $\varphi \sim \psi$ by T, as desired.

   (e) By a similar method to (d), use the limit definition of the derivative to show the contrapositive to (b).

Solution: Suppose V is G-invariant with respect to Φ. Then for all $X \in G$ and $v \in V$, $\Phi(X)v \in V$. In partiacular, for all $X \in \mathfrak{g}$ and $t \in \mathbb{R}$, $\Phi(e^{tX})v \in V$. Thus we have

$$\frac{\Phi(e^{tX}) - I}{t} \cdot v \in V$$

for all $t \neq 0$ by vector space axioms. Since $\Phi(e^{tX})$ is differentiable at $t=0$ and V is a topologically closed subspace of $\mathbb{C}^n$, we have that $\frac{\mathrm{d}}{\mathrm{d}t}\Phi(e^{tX})\Big|_{t=0} v \in V$. That is, $\varphi(X)v \in V$ for all X and v, showing V is $\mathfrak{g}$-invariant with respect to φ as desired.

(f) Let $G \leq \mathrm{GL}(\mathbb{C}^n)$ be a connected, compact matrix Lie group and $\mathfrak{g}$ be its Lie algebra. Let $\Phi\colon G \to \mathrm{GL}(V)$ be a finite-dimensional representation and $\varphi\colon \mathfrak{g} \to \mathsf{gl}(V)$ be its corresponding Lie algebra representation. Conclude that $\mathfrak{g}$ is completely reducible.

Solution: By worksheet 1, C4, Φ is completely reducible. That is, Φ is equivalent to a unitary representation $\Psi\colon G \to \mathrm{GL}(\mathbb{C}^n)$, and there exist G-invariant subspaces $V_1, \ldots, V_m$ with respect to Ψ such that $\Psi|_{V_i}$ is irreducible for all $1 \leq i \leq m$. Part (e) implies $V_1, \ldots, V_m$ are $\mathfrak{g}$-invariant with respect to φ and part (c) implies $\psi|_{V_i}$ is irreducible for all $1 \leq i \leq m$, where ψ is the Lie algebra representation corresponding to Ψ. We have that $\psi = \psi|_{V_1} \oplus \cdots \oplus \psi|_{V_m}$, and by part (d) we conclude that $\varphi \sim \psi$. Hence φ is completely reducible.

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